CRACK PROPAGATION IN CONTINUOUS MEDIA

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G.P. CHEREPANOV
(Moscow)

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The Griffith-Irwin theory is based upon the concept of surface tension of a solid, which is a physical constant of the material and depends only on temperature [1 and 2]. According to this theory, the stress intensity factor at the edge of a moving equilibrium tension crack in a linearly elastic body is a constant of the material. By introducing an effective surface energy density instead of surface tension, Irwin and Orowan [3] extended the criterion to materials that are not ideally brittle (the concept of quasi-brittle fracture). It is appropriate to remark here that the effective surface energy density is not a constant of the material, but generally depends on the rate of crack growth, the strain history, etc.

The Griffith-Irwin approach is applied in this paper to the study of crack development in an arbitrary continuous medium. This approach, in conjunction with the formulation of the singular problem of the "fine structure" at the edge of the crack, permits the derivation of a condition determining crack growth in an arbitrary continuous medium (Section 1). This leads to the results that are already known for a linearly elastic solid (Section 2).

The application of the general condition to certain plastic solids (Section 3) and to linearly viscoelastic solids (Section 4) is considered.

1. The condition of neutral equilibrium at the edge of a crack.

1.1. Let a continuous deformable body contain cracks which are surfaces of discontinuity of displacement. We shall consider the strains of the medium to be small. For definiteness we shall restrict our consideration to tension cracks with smooth surfaces, which, moreover, satisfy a condition of local symmetry. In accordance with this condition, in a small neighborhood of each point of the edge of a crack symmetry exists with respect to a plane tangent to the surface of the crack at that point. Without loss of generality it can also be considered that some vicinity of the crack surface near any point of the edge of the crack is free of traction.

Let us consider the neighborhood of an arbitrarily chosen point O of the edge of the crack which is small compared to a characteristic linear dimension of the body. We introduce a system of rectangular cartesian coordinates x, y, z with origin at the point O. Here the y axis is directed along the normal to the crack surface, the z axis is along the edge of the crack, and the x axis is directed into the body. The continuous medium in a small neighborhood of each point of the edge of the crack is in a state of plane strain. It is therefore possible to study the processes of deformation and fracture of the body in the small region near O with a two-dimensional picture (Fig. 1), regarding the crack as rectilinear, semi-infinite, and free of traction all along its length. Then in the whole region, infinity included, all the functions which characterize the stresses, displacements, temperature, etc. will be determined by the asymptotic behavior of the corresponding quantities in a small neighborhood of the point O of the original body ("the microscope principle"). We note that as a consequence of the condition of local symmetry, the normal displacement and the shearing traction along the extension of the plane of the crack (for $\infty > x > 0$) are zero.

We shall restrict ourselves to consideration of those processes which involve only mechanical and thermal energy. We denote by C a closed curve in the x-y plane encircling the point O (Fig. 1). The largest distance from an arbitrary point of the curve C to the origin O is small relative to the characteristic linear dimension of the body. In what follows the curve C may be considered to be a circle without loss of generality. The radius of the circle with center at O tends to infinity in the problem of "fine structure". Let us fix the

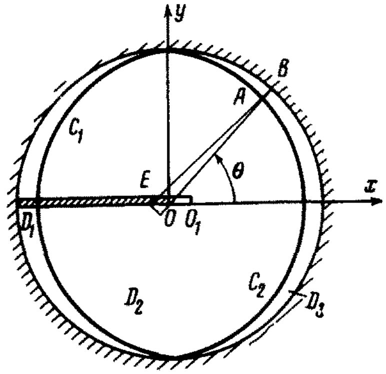

Fig. 1

curve C and study the process of straining and fracture of the medium D bounded by C. We denote by r and θ plane polar coordinates with pole at the point O; R is the radius of the circle C.

In accordance with the law of conservation of energy, the work $A^{\cdot}$ performed per unit time by the surface tractions on the curve C and by the body forces in D plus the thermal energy $Q^{\cdot}$ communicated to the body D per unit time across the curve C is equal to the rate of increase of the sum of the kinetic energy $K^{\cdot}$ and the internal energy $W^{\cdot} + \Pi^{\cdot}$ of the body in the region D, i.e.

$$A^{\cdot} + Q^{\cdot} = K^{\cdot} + W^{\cdot} + \Pi^{\cdot}$$

$$A^{\cdot} = R\int_0^{2\pi} [(\sigma_x \cos\theta + \tau_{xy}\sin\theta)\, u^{\cdot} + (\tau_{xy}\cos\theta + \sigma_y \sin\theta)\, v^{\cdot}]\, d\theta + \int_D \rho\,(F_x u^{\cdot} + F_y v^{\cdot})\, dx\, dy$$

$$Q^{\cdot} = R\int_0^{2\pi} (q_x^{\cdot}\cos\theta + q_y^{\cdot}\sin\theta)\, d\theta, \qquad \Pi^{\cdot} = 2\gamma l^{\cdot}$$

$$K^{\cdot} = \frac{1}{2}\frac{d}{dt}\int_D \rho\,(u^{\cdot 2} + v^{\cdot 2})\, dx\, dy, \qquad W^{\cdot} = \frac{d}{dt}\int_D \rho U\, dx\, dy \tag{1.1}$$

The following notation is introduced in these expressions: u and v are the components of the displacement vector along the x and y axes; σ_x, σ_y, τ_{xy}, and σ_z are the components of the stress tensor; $q_x^{\cdot}$ and $q_y^{\cdot}$ are the components of the heat flow vector; F_x and F_y are the components of the body forces; ρ and U are, respectively, the mass density and the internal energy per unit mass of an element of the material located at the point (x, y) at time t; γ is the surface energy per unit area (neglecting the latent heat of surface formation, it is equal to the surface tension of the material); $l^{\cdot}$ is the rate of propagation of the crack along the x axis. The dot on a symbol denotes a derivative with respect to time. The total internal energy of the body is obviously formed from the sum of the volume energy W and the surface energy Π.

All the functions which occur in (1.1) can in principle be found from the solution of the problem in the large for any model of a continuous body up to one undetermined constant which characterizes the position of the crack at the point O under consideration. Eq. (1.1) serves as a supplementary condition at the edge of the lengthening crack to determine this constant, and, thereby permits formulation of the problem of crack development in an arbitrary continuum.

We note that the length l of a propagating crack is a single-valued function of time t and may, therefore, be taken as the timelike variable in Eq. (1.1) if time is eliminated with the aid of the relation $l = l(t)$. The condition (1.1) then assumes the following form:

$$R\int_0^{2\pi} \left[(\sigma_x\cos\theta + \tau_{xy}\sin\theta)\frac{\partial u}{\partial l} + (\tau_{xy}\cos\theta + \sigma_y\sin\theta)\frac{\partial v}{\partial l}\right] d\theta + \frac{R}{l^{\cdot}}\int_0^{2\pi} (q_x^{\cdot}\cos\theta + q_y^{\cdot}\sin\theta)\, d\theta = \frac{1}{2}\, l^{\cdot 2}\frac{d}{dl}\int_D \rho\left[\left(\frac{\partial u}{\partial l}\right)^2 + \left(\frac{\partial v}{\partial l}\right)^2\right] dx\, dy + 2\gamma + \frac{d}{dl}\int_D \rho U\, dx\, dy - \int_D \rho\left(F_x\frac{\partial u}{\partial l} + F_y\frac{\partial v}{\partial l}\right) dx\, dy \tag{1.2}$$

The derivatives with respect to l may clearly be computed from the singular solution in the following way:

$$\frac{\partial f}{\partial l} = \lim_{\Delta l \to 0} \frac{f(x, y, l + \Delta l) - f(x, y, l)}{\Delta l} \tag{1.3}$$

where the function $f(x, y, l)$ is known from the singular solution and corresponds to the position of the edge of the crack at point O.

The function $f(x, y, l + \Delta l)$ corresponds to the position of the end of the crack at the point O_1, which is displaced by the distance Δl along the x axis (Fig. 1).

The supplementary condition (1.2) at the edge of the crack will play an important part in what follows. It should be emphasized that in the general case all the terms in Eq. (1.2) have the same order. However, for various media and regimes of crack development they may play different roles. For instance, in the case of a quasi-static crack, the rate of growth of which is considerably slower than the sound speed, the first term on the right-hand side of Eq. (1.2), representing the kinetic energy, can be neglected.

1.2. Let us transform the condition (1.2) into a more convenient form. Let the singular solution corresponding to the position of the edge of the crack at point O have the form

$$u = u_0(N, x, y), \; v = v_0(N, x, y), \; \sigma_x = \sigma_{x0}(N, x, y), \ldots, \rho U = \rho U_0(N, x, y) \tag{1.4}$$

Here N is an undetermined constant; it can be found only from the solution of the problem in the large and is some function of the shape of the body (in particular, of the length l of the crack) and of the parameters of the external loading. In dynamic problems N also depends on the parameters which define $l(t)$ (e.g. on the crack velocity in the case of constant velocity crack growth).

In the course of development of the crack, let the edge of the crack reach the point O_1 at a distance Δl from the original position at point O (Fig. 1). Then the function of the parameters of the problem N is incremented by ΔN. On the basis of (1.4), the singular solution which corresponds to the edge of the crack at O_1 has the form

$$u = u_0(N + \Delta N, \; x - \Delta l, \; y), \quad v = v_0(N + \Delta N, \; x - \Delta l, \; y)$$

$$\sigma_x = \sigma_{x0}(N + \Delta N, \; x - \Delta l, \; y), \ldots, \quad \rho U = \rho U_0(N + \Delta N, \; x - \Delta l, \; y) \tag{1.5}$$

The law of conservation of energy (1.1) for the region D enclosed by the circle C can be written in the form

$$\lim_{\Delta N \to 0} \left(\frac{\Delta A}{\Delta N} + \frac{\Delta Q}{\Delta N} - \frac{\Delta K}{\Delta N} - \frac{\Delta W}{\Delta N} \right)_{\Delta l = 0} \Delta N +$$

$$+ \lim_{\Delta l \to 0} \left(\frac{\Delta A}{\Delta l} + \frac{\Delta Q}{\Delta l} - \frac{\Delta K}{\Delta l} - \frac{\Delta V}{\Delta l} - 2\gamma \right)_{\Delta N = 0} \Delta l = 0 \tag{1.6}$$

where Δx denotes an increment in the quantity x. The coefficient of ΔN in (1.6) equals zero by virtue of the law of conservation of energy for a stationary crack. It follows from this that in the condition (1.2) the derivatives with respect to l must be computed for $N =$ $=$ const. Let us draw a semicircle C_1 of radius R with center at O_1 and a semicircle C_2 also of radius R with center at E $(x = -\Delta l, y = 0)$ (Fig. 1). The parts of the region D included between C and C_1, C_1 and C_2, C_2 and C will be denoted by D_1, D_2, and D_3, respectively $(D = D_1 + D_2 + D_3)$. For $N =$ const the following Eq. holds up to small quantities of higher order $(\Delta l \ll R)$:

$$\int_{D_2 + D_3} \rho U |_{O_1} \, dx \, dy = \int_{D_1 + D_2} \rho U |_{O} dx \, dy \tag{1.7}$$

Here the values of the integrands are taken at the position of the edge of the crack indicated after the vertical line. Using (1.7) and the relation $AB = \Delta l \cos \theta$ (Fig. 1), we find

$$\frac{d}{dl} \int_D \rho U \, dx \, dy = \lim_{\Delta l \to 0} \frac{1}{\Delta l} \Big[\int_{D_1 + D_2 + D_3} \rho U |_{O_1} \, dx \, dy - \int_{D_1 + D_2 + D_3} \rho U |_{O} \, dx \, dy \Big] =$$

$$= -R \int_0^{2\pi} \rho U_0 \cos \theta \, d\theta \tag{1.8}$$

In an entirely analogous manner we calculate the derivative of the kinetic energy in the region D with respect to l for N = const

$$\frac{dK}{dl} = -\frac{1}{2} R l^{\cdot 2} \int_0^{2\pi} \rho \left[(\partial u / \partial l)^2 + (\partial v / \partial l)^2\right] \cos\theta \, d\theta \tag{1.9}$$

With the aid of Eqs. (1.4) and (1.5), we obtain

$$\left(\frac{\partial u}{\partial l}\right)_{N=\text{const}} = -\frac{\partial u_0}{\partial x}, \qquad \left(\frac{\partial v}{\partial l}\right)_{N=\text{const}} = -\frac{\partial v_0}{\partial x} \tag{1.10}$$

The work of the body forces may be expressed in the following form, after making use of (1.8) and (1.10)

$$\int_D \rho \left(F_x \frac{\partial u}{\partial l} + F_y \frac{\partial v}{\partial l}\right) dx\, dy = \int_D \frac{d(\rho H)}{dl}\, dx\, dy =$$

$$= \frac{d}{dl} \int_D \rho H \, dx\, dy = -R \int_0^{2\pi} \rho H \cos\theta \, d\theta \tag{1.11}$$

$$\rho H = \int \rho \left(F_x \frac{\partial u}{\partial x} + F_y \frac{\partial v}{\partial x}\right) dx$$

Combining (1.8) to (1.11), we may write the condition (1.2) in the form

$$R \int_0^{2\pi} \left[(\rho U + K_* - \rho H) \cos\theta + \frac{1}{l^{\cdot}} (q_x^{\cdot} \cos\theta + q_y \sin\theta) - A_*\right] d\theta = 2\gamma$$

$$K_* = {}^1/_2 \rho l^{\cdot 2} \left[(\partial u / \partial x)^2 + (\partial v / \partial x)^2\right] \tag{1.12}$$

$$A_* = (\sigma_x \cos\theta + \tau_{xy} \sin\theta)(\partial u / \partial x) + (\tau_{xy} \cos\theta + \sigma_y \sin\theta)(\partial v / \partial x)$$

In this expression all the functions are computed directly from the singular solution corresponding to the position of the edge of the crack at the point O (the subscript o is dropped for simplicity). Each term in the integrand of (1.12) must have a singularity at the edge of the crack of the type $1/r$ in order that it make a finite contribution to the sum. A singularity of higher order $r^{-\lambda}$ ($\lambda > 1$) is not permissible, since this would lead to a violation of the law of conservation of energy (1.12) (we recall that in the singular solution under consideration there is no characteristic linear dimension). Terms with a singularity of lower order $r^{-\lambda}$ ($\lambda < 1$) will obviously drop out of Eq. (1.12). Another interesting conclusion follows from Eq. (1.12); the role of the dissipative term (the third expression) becomes less important as the speed of the crack increases, while the significance of the dynamical term (the second expression) grows.

The condition of limiting equilibrium at the crack tip (1.12) can also be presented in another convenient form in which only the displacement and stress fields in the vicinity of the edge of the crack appear. At each point of the continuous medium occupying the region D, the local law of conservation of energy [4] is satisfied

$$\sigma_x \varepsilon_x^{\cdot} + \sigma_y \varepsilon_y^{\cdot} + 2\tau_{xy} \varepsilon_{xy}^{\cdot} + \partial q_x^{\cdot} / \partial x + \partial q_y / \partial y = \rho U^{\cdot} \tag{1.13}$$

With the aid of the divergence theorem and of Eqs. (1.13) and (1.8), we carry out the following transformations

$$\frac{R}{l^{\cdot}} \int_0^{2\pi} (q_x^{\cdot} \cos\theta + q_y^{\cdot} \sin\theta)\, d\theta = \frac{1}{l^{\cdot}} \int_D \left(\frac{\partial q_x^{\cdot}}{\partial x} + \frac{\partial q_y^{\cdot}}{\partial y}\right) dx\, dy =$$

$$= \frac{1}{l^{\cdot}} \int_D \left[\rho U^{\cdot} - (\sigma_x \varepsilon_x^{\cdot} + \sigma_y \varepsilon_y^{\cdot} + 2\tau_{xy} \varepsilon_{xy}^{\cdot})\right] dx\, dy =$$

$$= -R \int_0^{2\pi} \rho U \cos\theta \, d\theta + R \int_0^{2\pi} \mathcal{E} \cos\theta \, d\theta$$

$$\mathcal{E} = \int \sigma_x d\varepsilon_x + \sigma_y d\varepsilon_y + 2\tau_{xy} d\varepsilon_{xy} \tag{1.14}$$

Then, applying (1.14), we may write Eq. (1.12) as

$$R \int_0^{2\pi} [(\partial + K_* - \rho H) \cos\theta - A_*] \, d\theta = 2\gamma \tag{1.15}$$

in which the notation ∂, K_*, ρH and A_* is explained in the expressions of (1.14). Physically, the terms ∂, K_*, ρH and A_* are, respectively, the work of the internal forces, the kinetic energy, the work of the body forces, and the work of the surface tractions.

We should emphasize that in the derivation of the limiting condition at the edge of the crack no appeal has been made to the mechanical properties of the continuum. Only the fact of continuity of the medium has been used. Obviously, if the left-hand side of Eq. (1.12) or (1.15) is smaller than 2γ, the crack will not grow.

The approach which has been applied can be generalized in a direction analogous to the concept of quasi-brittle fracture [3]. The quantity γ must then be interpreted as the work of the irreversible deformations in the vicinity of the edge of the crack, which is not accounted for in the model which has been adopted.

2. Elastic body.

2.1. We shall first examine the problem of isothermal development of quasi-static tension cracks in a homogeneous, isotropic material which is linearly elastic up to fracture. Any conversion between mechanical and thermal energy will be neglected.

The singular solution for a semi-infinite tension crack for an isothermal process can be found by the Kolosov-Muskhelishvili method [5]

$$\sigma_x + \sigma_y = 2Nr^{-1/2} \cos (1/2\theta), \qquad \sigma_z = 2\nu N r^{-1/2} \cos (1/2\theta)$$

$$\sigma_x - i\tau_{xy} = 1/4\, N r^{-1/2} [e^{-1/2 i\theta} + 2e^{1/2 i\theta} + e^{-5/2 i\theta}]$$

$$\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{(1+\nu)N}{4E\sqrt{r}} [e^{5/2 i\theta} - 3 e^{1/2 i\theta} + 2\,(3-4\nu)\, e^{-1/2 i\theta}] \tag{2.1}$$

where N is the stress intensity factor, E is Young's modulus, and ν is Poisson's ratio. The stress intensity factor is determined from the solution of the problem in the large and is some function of the shape of the body and the boundary conditions.

Under the assumptions which have been made, the condition (1.12) has the form for elastic bodies

$$R \int_0^{2\pi} \Big[\rho U \cos\theta - (\sigma_x \cos\theta + \tau_{xy} \sin\theta) \frac{\partial u}{\partial x} - - (\tau_{xy} \cos\theta + \sigma_y \sin\theta) \frac{\partial v}{\partial x}\Big] d\theta = 2\gamma \tag{2.2}$$

The internal energy per unit volume is equal to the strain energy density

$$\rho U = 1/2\,(1-\nu^2)\, E^{-1} (\sigma_x + \sigma_y)^2 + (1+\nu)\, E^{-1} (\tau_{xy}^2 - \sigma_x \sigma_y) \tag{2.3}$$

Substituting the functions from (2.1) and (2.3) into Eq. (2.2) and computing the integrals, we find

$$\pi N^2 = E\,\gamma\, (1-\nu^2)^{-1} \tag{2.4}$$

This is Irwin's well-known result [3] which determines the condition of limiting equilibrium at the edge of a tension crack in a linearly elastic body. We remark that Irwin's method [3] is applicable only for linearly elastic bodies.

2.2. We now turn to the nonlinearly elastic, homogeneous, isotropic material having a tension crack. We shall consider that the medium is incompressible and obeys an arbitrary power-law relation between the shear stress intensity J and the shear strain intensity Γ. This dependence can serve as a convenient approximation for an arbitrary relation between J and Γ in the range of magnitudes which are characteristic of the vicinity of a crack tip. It can be shown that in this case of a power law, the variables in the equations of the theory of elasticity are separable, at least in cartesian and plane polar coordinates.

The basic equations of the problem are [6]:

Eqs. of equilibrium

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} = 0, \qquad \frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_\theta}{\partial \theta} + 2\frac{\tau_{r\theta}}{r} = 0 \tag{2.5}$$

the compatibility Eq.

$$2\frac{\partial}{\partial r}\left(r\frac{\partial\varepsilon_{r\theta}}{\partial\theta}\right)=\frac{\partial^2\varepsilon_r}{\partial\theta^2}-r\frac{\partial\varepsilon_r}{\partial r}+r\frac{\partial^2(r\varepsilon_\theta)}{\partial r^2} \tag{2.6}$$

and the stress-strain relations

$$\varepsilon_r=-\varepsilon_\theta={}^1/_2 aJ^{\varkappa}(\sigma_r-\sigma_\theta),\quad \varepsilon_{r\theta}=aJ^{\varkappa}\tau_{r\theta} \tag{2.7}$$

$$\Gamma=2aJ^{\varkappa+1},\quad 2J=\sqrt{(\sigma_r-\sigma_\theta)^2+4\tau_{r\theta}^2},\quad \Gamma=2\sqrt{\varepsilon_r^2+\varepsilon_{r\theta}^2}$$

where a and $\varkappa$ are the elastic constants.

We seek the singular solution for the problem in the form:

$$\sigma_r=-(\lambda+1)^{-1}r^{\lambda}[f''(\theta)+(\lambda+2)f(\theta)]$$

$$\sigma_\theta=-(\lambda+2)r^{\lambda}f(\theta),\qquad \tau_{r\theta}=r^{\lambda}f'(\theta) \tag{2.8}$$

where $f(\theta)$ is an arbitrary function, and λ is an undetermined constant. It is easy to verify that the equilibrium Eqs. (2.5) are then satisfied. We substitute Expressions of (2.8) into Eqs. (2.7) for the strains, and then substitute the results into the Eq. of compatibility (2.6). We finally obtain the following ordinary differential equation for the determination of the function $f(\theta)$:

$$2[\lambda(\varkappa+1)+1](f'\Phi^{\varkappa})'=$$

$$=[d^2/d\theta^2-2\lambda(\varkappa+1)-\lambda^2(\varkappa+1)^2][\Phi^{\varkappa}\sqrt{\Phi^2-(f')^2}] \tag{2.9}$$

$$4\Phi^2=4(f')^2+[(\lambda+2)f-(\lambda+1)^{-1}(f''+\lambda f+2f)]^2$$

The sides of the crack are free of traction; this leads, in accordance to (2.8) to the boundary conditions

$$f(\theta)=f'(\theta)=0\quad(\theta=\pm\pi) \tag{2.10}$$

The solution of the differential equation (2.9) subject to the boundary conditions (2.10) is a problem of numerical analysis. It is curious that the constant λ which determines the character of the stress distribution near the edge of the crack and which is a peculiar type of eigenvalue of the boundary-value problem (2.9), (2.10), can be found from physical considerations.

For on the basis of the condition (2.2) which is also obviously valid in this case, the integrand in (2.2) must have a singularity of the type $1/r$ at the crack tip.

In the case under study, the internal energy density is equal to the strain energy density

$$\rho U=2a(\varkappa+1)(\varkappa+2)^{-1}J^{\varkappa+2} \tag{2.11}$$

We find from this and Eqs. (2.7) and (2.8)

$$\lambda=-1/(\varkappa+2) \tag{2.12}$$

In particular, in the linear problem, $\varkappa=0$ and $\lambda=-\frac{1}{2}$, as was given above in Eqs. (2.1).

Thus the stresses near a crack tip in a material obeying a power law have a singularity of the form $r^{-1/(\varkappa+2)}$. The function $f(\theta)$ is found from the boundary-value problem (2.9), (2.10) up to an undetermined factor (if symmetry is also specified). The constant factor is completely determined by the supplementary condition (2.2).

3. Plastic body. We shall now concern ourselves with the case of an elastic-perfectly plastic material. We shall assume that the curve which separates the elastic and plastic regions in the plane $z=$ ^ (Fig. 1) entirely surrounds the edge of the tension crack and that unloading does not take place anywhere in the plastic region. The stresses near the edge of the crack can then be found using only the equilibrium equations and the Mises yield condition $J=\tau_s$, where τ_s is the yield stress in shear.

Let the assumed shear line field have the shape depicted in Fig. 2. We then have [7 and 8]

$$\sigma_x=\pi\tau_s,\quad \sigma_y=(2+\pi)\tau_s,\quad \tau_{xy}=0\quad({}^1/_4\pi>|\theta|>0)$$

$$\sigma_x=\sigma-\tau_s\sin 2\theta,\quad \sigma_y=\sigma+\tau_s\sin 2\theta \tag{3.1}$$

$$\tau_{xy}=\tau_s\cos 2\theta,\quad \sigma=(1+{}^3/_2\pi-2\theta)\tau_s\quad({}^3/_4\pi>|\theta|>{}^1/_4\pi)$$

$$\sigma_x=\tau_s,\qquad =\tau_{xy}=0\quad(\pi>|\theta|>{}^3/_4\pi)$$

From the Prandtl-Reuss equation and the condition of incompressibility we determine the velocity field in the vicinity of the edge of the crack

$$v^{\cdot} = f_1^{\cdot}(x-y) - f_1^{\cdot}(x+y), \quad u^{\cdot} = f_1^{\cdot}(x-y) + f_1^{\cdot}(x+y) \quad (^1/_4\pi > |\theta| > 0)$$

$$u_r^{\cdot} = f_2^{\cdot\prime}(\theta) \quad u_\theta^{\cdot} = f_3^{\cdot}(r) - f_2^{\cdot}(\theta) \quad (^3/_4\pi > |\theta| > {}^1/_4\pi) \tag{3.2}$$

$$u^{\cdot} = f_4^{\cdot}(x-y) + f_5^{\cdot}(x+y), \quad v^{\cdot} = f_4^{\cdot}(x-y) - f_5^{\cdot}(x+y) \quad (\pi > |\theta| > {}^3/_4\pi)$$

Here $f_1^{\cdot}$, $f_2^{\cdot}$, $f_3^{\cdot}$ and $f_4^{\cdot}$ are arbitrary functions; $u_r^{\cdot}$ and $u_\theta^{\cdot}$ are the components of the velocity vector in the r and θ directions. From physical considerations we require that the velocities be bounded in the vicinity of the edge of the crack. The displacements near the edge of the crack corresponding to Eqs. (3.2) cannot be written in the form (1.4). Therefore, it is not in general possible to use (1.12).

We write the local law of conservation of energy which is satisfied at each point of the plastic region of the elastic-perfectly plastic body for the case of plane strain [4]

$$\tau_s\Gamma^{\cdot} + \partial q_x^{\cdot} / \partial x + \partial q_y^{\cdot} / \partial y = \rho U^{\cdot} \tag{3.3}$$

where Γ is the intensity of shear strain. With the aid of Eq. (3.3) and the divergence theorem, Eq. (1.2) can be written in the following form in the absence of body and inertia forces:

$$R\int_0^{2\pi}\left[(\sigma_x\cos\theta + \tau_{xy}\sin\theta)\frac{\partial u}{\partial l} + (\tau_{xy}\cos\theta + \sigma_y\sin\theta)\frac{\partial v}{\partial l}\right]d\theta = \tau_s\int_D \frac{\partial\Gamma}{\partial l}\,dx\,dy \tag{3.4}$$

where D is a circular region lying wholly within the plastic zone.

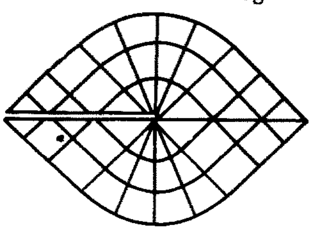

Fig. 2

We shall examine the most important special case, that of a plastic region whose characteristic linear dimensions are small compared to the characteristic linear dimension of the body. The stresses and strains in the plastic region (and also the form and dimensions of the plastic region) will then depend only on a single parameter N which is the stress intensity factor near the edge of a crack as computed from the purely elastic problem under the assumption of no plastic region. In this case the method given above in Section 2 par. 2.2 can be applied, and the following condition can be obtained from Eq. (3.4):

$$R\tau_s\int_0^{2\pi}\Gamma\cos\theta\,d\theta - R\int_0^{2\pi}\left[(\sigma_x\cos\theta + \tau_{xy}\sin\theta)\frac{\partial u}{\partial x} + \right.$$

$$\left. + (\tau_{xy}\cos\theta + \sigma_y\sin\theta)\frac{\partial v}{\partial x}\right]d\theta = 2\gamma \tag{3.5}$$

By analyzing Eqs. (3.2) which give the velocities near the edge of the crack, we can conclude from the boundedness of the velocities that only the function $f_2(\theta)$ provides a finite contribution in Eq. (3.5). The condition (3.5) can finally be written as:

$$2R\tau_s\int_{\pi/4}^{3\pi/4}\varepsilon_{r\theta}\cos\theta\,d\theta - R\int_{\pi/4}^{3\pi/4}\left(\sigma_r\frac{\partial u_r}{\partial x} + \tau_{r\theta}\frac{\partial u_\theta}{\partial x}\right)d\theta = \gamma \tag{3.6}$$

Substituting the stresses and displacements of (3.1) and (3.2) into (3.6), we find

$$F = F_0 + \gamma / \tau_s \tag{3.7}$$

$$F = \int_{\pi/4}^{3\pi/4}\{[f_2''(\theta) + f_2(\theta)]\cos\theta + (1 + {}^3/_2\pi - 2\theta)f_2''(\theta)\sin\theta - f_2'(\theta)\sin\theta\}\,d\theta$$

where the function F_0 is equal to F at the initial instant of loading. It characterizes the initial (residual) strains.

The condition (3.7) serves to determine the parameter N arising in the function $f_2(\theta)$, and thereby permits us to formulate the problem of the development of tension cracks in incompressible elastic-plastic bodies. It is assumed in Eqs. (3.6) and (3.7) that the radius R of the region D is considerably smaller than the characteristic linear dimension of the plastic region L ($R \ll L$). We should note that a rigorous development of the concept of quasi-brittle fracture can be obtained only by combining the approach just considered with that applied in Section 2, which corresponds to $R \gg L$.

4. **Viscoelastic body.**

4.1. Let us consider an isotropic homogeneous viscoelastic body with tension cracks. The process of crack propagation will be considered as quasi-static and isothermal. The stress-strain relation for this case can be represented in the most general form [9] as

$$\varepsilon_x = \mathrm{E}^{-1}\sigma_x - \mathrm{E}^{-1}\nu(\sigma_y + \sigma_z), \qquad \varepsilon_y = \mathrm{E}^{-1}\sigma_y - \mathrm{E}^{-1}\nu(\sigma_x + \sigma_z)$$

$$\varepsilon_{xy} = \mathrm{E}^{-1}(1+\nu)\tau_{xy}, \qquad \mathrm{E}^{-1}[\sigma_z - \nu(\sigma_x + \sigma_y)] = 0 \tag{4.1}$$

where E^{-1} and ν are commutative linear operators involving time t which have the form

$$\mathrm{E}^{-1}f = \int_0^t E_0(t-\tau) f(\tau)\, d\tau, \qquad \nu f = \int_0^t \nu_0(t-\tau) f(\tau)\, d\tau \tag{4.2}$$

The functions $E_0(x)$ and $\nu_0(x)$ belong to the class of generalized functions. The relations (4.1) are written for conditions of plane strain, which obtain in a small neighborhood of each point of the smooth edge of a crack.

Problems of determining the stress and strain fields in visco-elastic solids with propagating cracks whose edges move with time are extraordinarily difficult for boundary conditions of general type. However, in one interesting case of plane strain a remarkable situation occurs: the stresses σ_x, σ_y and τ_{xy} in a finite, simply-connected body made of a viselastic material and having moving cracks are the same as in the corresponding elastic problem provided that only traction boundary conditions are applied. This analogy also holds in the case of an infinite or multiply connected body if the resultant force and moment of the external tractions applied to each of the bounding contours are individually equal to zero. The analogy is easily proved by writing the compatibility conditions for the transformed strains (using Laplace transforms) $\nabla^2(\sigma_x^\circ + \sigma_y^\circ) = 0$, shifting to the original quantities to get $\nabla^2(\sigma_x + \sigma_y) = 0$, and recalling the well-known results of the two-dimensional theory of elasticity on the independence of the state of stress on the elastic constants.

In particular, it can be shown that the singular problem for the stress and strain fields near the edge of a crack has the following solution by analogy with Section 2:

$$\sigma_x + \sigma_y = 2Nr^{-1/2}\cos(\theta/2), \qquad \sigma_z = 2\nu N r^{-1/2}\cos(\theta/2)$$

$$\sigma_x - i\tau_{xy} = {}^1/_4 N r^{-1/2}[e^{-i\theta/2} + 2e^{i\theta/2} + e^{-5i\theta/2}]$$

$$\partial u/\partial x + i\partial v/\partial x = {}^1/_4 r^{-1/2}\mathrm{E}^{-1}(1+\nu)[2(3-4\nu)\, N e^{-i\theta/2} + $$
$$+ N e^{5i\theta/2} - 3Ne^{i\theta/2}] \tag{4.3}$$

Here the stress intensity factor, which in Eqs. (2.1) was a function of the boundary conditions and the shape of the body, is generally a function of time as well. This function is determined from the solution of the problem in the large. If the analogy presented above holds then obviously there will be no explicit dependence of N upon time.

For the case under consideration, when there are no body forces the general condition at the edge of the crack can be written in the form

$$R\int_0^{2\pi} (\mathcal{E}\cos\theta - A_*)\, d\theta = 2\gamma \tag{4.4}$$

Using Eqs. (4.1), (4.3), (1.12), and (1.14), we carry out the following calculations:

$$\mathcal{E} = \int_0^t (\sigma_x \varepsilon^{\cdot}_x + \sigma_y \varepsilon^{\cdot}_y + 2\tau_{xy}\varepsilon^{\cdot}_{xy})\, dt = \int_0^t [\sigma_x \mathrm{E}^{-1}(1+\nu)\sigma^{\cdot}_x + \sigma_y \mathrm{E}^{-1}(1+\nu)\sigma^{\cdot}_y -$$
$$- (\sigma_x + \sigma_y)\mathrm{E}^{-1}(1+\nu)\nu(\sigma^{\cdot}_x + \sigma^{\cdot}_y) + 2\tau_{xy}\mathrm{E}^{-1}(1+\nu)\tau^{\cdot}_{xy}]\, dt$$

$$R\int_0^{2\pi} \mathcal{E}\cos\theta d\theta = \pi\int_0^t N\mathrm{E}^{-1}(1+\nu)(1-2\nu)N^{\cdot}dt \tag{4.5}$$

$$R\int_0^{2\pi} A_* d\theta = \mathrm{Im}\oint (\sigma_x - i\tau_{xy})(\partial u/dx + i\partial v/\partial x)\, dz + R\int_0^{2\pi}(\sigma_x + \sigma_y)\sin\theta\,(\partial v/\partial x)\, d\theta =$$
$$= -{}^1/_4 \pi N \mathrm{E}^{-1}(1+\nu)(3-2\nu)N \qquad (z = Re^{i\theta})$$

where the contour integrals are taken in the counterclockwise direction.

Finally, the condition of limiting equilibrium at the edge of a tension crack can be written in accordance with (4.4) and (4.5) as

$$2\int_0^t NE^{-1}(1+\nu)(1-2\nu)N^{\cdot}dt + NE^{-1}(1+\nu)(3-2\nu)N = \frac{4\gamma}{\pi} \quad (4.6)$$

The condition (4.6) is generally, as is easy to see, a nonlinear integro-differential equation which serves to determine the function $N(t)$ and, thereby, the rule governing the propagation of a crack with time.

In the case of an elastic body, when the operators $\mathbf{E}^{-1}$ and ν are elastic constants (the reciprocal of Young's modulus and Poisson's ratio), it can readily be seen that Irwin's condition (2.4) is obtained from (4.6).

4.2. As an illustrative example let us consider an incompressible generalized linear solid [4]. The $\sigma - \varepsilon$ relation for stretching of a bar made of this material has the form

$$\varepsilon^{\cdot} + \zeta\varepsilon = (3\mu)^{-1}\sigma^{\cdot} + (3\eta)^{-1}\sigma \quad (4.7)$$

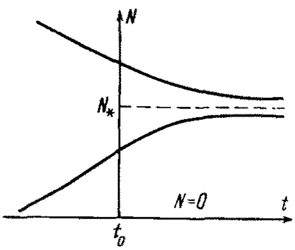

Fig. 3

where μ is the shear modulus, η is the viscosity coefficient, and ζ is some material constant.

It is not difficult to show that the kernel $E_0(x)$ of the operator $\mathbf{E}^{-1}$ corresponding to the material of Eq. (4.7) is

$$E_0(x) = (3\mu)^{-1}\delta(x) + 1/3(\eta^{-1} - \zeta\mu^{-1})\exp(-\zeta x) \quad (4.8)$$

where $\delta(x)$ is the delta-function defined by

$$\int_0^t \delta(t-\tau)\sigma(\tau)\,d\tau = \sigma(t)$$

For incompressible materials, the condition (4.6) has the following very simple form:

$$NE^{-1}N = (4\gamma)/(3\pi) \quad (4.9)$$

The nonlinear integral Eq. (4.9) with the kernel (4.8) reduces to the following differential Eq.:

$$-4\gamma\mu\eta N^{\cdot} + 4\gamma\zeta\mu\eta N = \pi\eta N^2 N^{\cdot} + \pi\mu N^3 \quad (4.10)$$

We find the solution of Eq. (4.10)

$$\frac{\pi N^2 - 4\gamma\eta\zeta}{\pi N_0^2 - 4\gamma\eta\zeta} = \left(\frac{N}{N_0}\right)^{\lambda} e^{-\zeta\lambda(t-t_0)} \quad \text{for} \quad t = t_0, \quad N = N_0\left(\lambda = \frac{2\mu}{\mu+\zeta\eta}\right) \quad (4.11)$$

The curve $N = N(t)$ corresponding to (4.11) has two asymptotes, $N = 0$ and $N = N_*$ (Fig. 3)

$$N_*^2 = 4\pi^{-1}\gamma\eta\zeta \quad (4.12)$$

The derivative $N^{\cdot}(t)$ which equals

$$N^{\cdot} = \frac{\mu N(4\gamma\eta\zeta - \pi N^2)}{\eta(\pi N^2 + 4\gamma\mu)} \quad (4.13)$$

is always negative for $N > N_*$ and is always positive for $0 < N < N_*$. Therefore the curve $N(t)$ consists of two parts, a monotonously increasing one ($0 < N < N_*$) and a monotonously decreasing one ($N > N_*$).

In the special case of a Maxwell solid, which is obtained from Eq. (4.7) by letting $\zeta \to 0$ for finite μ and η, Eq (4.11) assumes the form

$$t - t_0 + \frac{\eta}{\mu}\ln\frac{N}{N_0} = -\frac{2}{\pi}\eta\gamma\frac{N^2 - N_0^2}{N^2 N_0^2} \quad (4.14)$$

For the Maxwell solid $N_* = 0$ and the increasing branch of the curve disappears; N decreases monotonously with time and tends to zero for $t \to \infty$ (for $t \to -\infty$, $N \to \infty$).

For the case of a Kelvin solid

$$\sigma = 3\eta\varepsilon^{\cdot} + 3\mu\varepsilon \quad (4.15)$$

we obtain in precisely the same way

$$\frac{N^2}{N_0^2} = \frac{4\gamma\mu - \pi N^2}{4\gamma\mu - \pi N_0^2} e^{\frac{\mu}{\eta}(t-t_0)} \quad (4.16)$$

Here the curve $N(t)$ has qualitatively the same shape as in the case of the generalized linear solid (Fig. 3), but the magnitude of N_*^2 equals $4\gamma\mu/\pi$.

Let us now see how an isolated rectilinear crack of length $2l$ develops in an infinite medium in two cases: (1) under a constant tensile stress p in the far field (directed perpendicular to the line of the crack), and (2) with two equal and opposite concentrated forces P applied to the opposite sides of the crack at its center. On the basis of the analogy given in Section 4 par. 4.1, the stress intensity factors in the two cases are

$$N^2 = {}^1/_2 p^2 l, \qquad N^2 = {}^1/_2 \pi^{-2} P^2 l^{-1} \tag{4.17}$$

In the first case, if $N_0 < N_*$ at the initial instant of time, the crack grows, attaining (for p = const, as time approaches infinity) a critical length $l_* = 2N_*^2 p^{-2}$ (for a Kelvin or generalized linear solid). If, however, at the initial time $N_0 > N_*$ the crack grows dynamically, inasmuch as the equilibrium state is unstable (as p increases the length l of the crack decreases from (4.17)). In a Maxwell solid the equilibrium state will be unstable for all p.

In the second case, if at the initial time $N_0 < N_*$ the crack will not develop (or will become smaller for reversible cracking until the equilibrium value $l_* = ½\pi^{-2} P^2 N_*^{-2}$ is reached). If at the initial time $N_0 > N_*$ the crack grows stably (for an infinitely long time), tending to the equilibrium value l_*. For a Maxwell material the crack grows stably to infinite length for any initial N_0.

It is easy to see that in the general case if characteristic times of loading large compared to the relaxation time $\theta = \mu / \eta$ are considered, a Maxwell solid will fracture under any finite load, but cracks in a linear generalized or Kelvin solid will develop in exactly the same way as in a linearly elastic solid (if the conditions for the analogy are satisfied), provided that the limiting value of N coincides with the corresponding limiting value for the incompressible linearly elastic solid. This limiting value is equal to $2(\gamma\mu/\pi)^{½}$ for a Kelvin solid and $2(\gamma\eta\zeta/\pi)^{½}$ for the generalized linear solid.

It may be noted that the study of the kinetics of crack growth was undertaken by Kachanov [10] and Williams [11], who applied a less rigorous approach.

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